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DIFFERENTIAL EQUATIONS IN SPACES OF HILBERT SPACE VALUED DISTRIBUTIONS

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A Gaussian measure is introduced on the space of Hilbert space valued tempered distributions. It is used to define a Hilbert space valued Q-Wiener process and a white noise process with a nuclear covariance operator Q. The proposed construction is used for solving operator-differential equations with additive noise with the operator coefficient generating an n-times integrated exponentially bounded semigroup.

1. Introduction

Let X and Y be separable Hilbert spaces. We denote by $D'(X)$ the space of X-valued distributions defined on D, the space of infinitely differentiable functions with compact supports. By $D'_+(X)$ we denote the subspace of distributions from $D'(X)$ with supports bounded from below.

Any linear time-invariant dynamic system is fully determined by its state equation which can be written in the form

$$P * U = F, \tag{1}$$

where $P \in D'_+\big(\mathcal{L}(X;Y)\big)$, $U \in D'_+(X)$, $F \in D'_+(Y)$ (see [**1**]). The system is said to be invertible if there exists $G \in D'_+\big(\mathcal{L}(Y;X)\big)$, the convolution inverse for P, so that the equalities $P * G = \delta \otimes I_Y$ and $G * P = \delta \otimes I_X$ hold. In this case formula $U = G * F$ yields the unique solution of (1) (see details in [**1**]).

One can model stochastic influence of the environment on the system by introducing an appropriately defined 'noise' term W into the right-hand side of (1).

$$P * U = F + W. \tag{2}$$

A solution of the perturbed equation formally can be written in the form $U = Q*(F+W)$.

In this note we construct a Gaussian measure on the space of H-valued tempered distributions, where H is a separable Hilbert space, using the approach of [**3**]. We use the approach of [**2**] to define Q-Wiener process and Q-white noise process as generalised processes with values in H (where $Q : H \to H$ is a nuclear operator). This makes convolution $Q * BW$ well-defined for any linear bounded operator $B : H \to Y$ in the same sense as it is defined for Hilbert space valued distributions.

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2. Preliminaries

Consider a Gelfand triple

$$S \subseteq S_0 \subseteq S',$$

where $S_0 = L^2(\mathbb{R})$, S is the Schwartz space of rapidly decreasing test functions and S' is the space of corresponding tempered distributions.

Denote by $(\cdot,\cdot)_0$ and $|\cdot|_0$ the inner product and the corresponding norm in S_0. Consider the linear operator $A := -(d^2/dx^2)+x^2+1$. For all $p \in \mathbb{Z}$, $\xi \in S$ let $|\xi|_p = |A^p\xi|_0$. Let $(\cdot,\cdot)_p$ be the corresponding inner product and S_p be the completion of S with respect to $|\cdot|_p$. The space S_{-p} is the dual of S_p for each $p > 0$. Then we have the following inclusions:

$$S = \bigcap_{p\in\mathbf{N}} S_p \subset \cdots \subset S_{p+1} \subset S_p \subset \cdots \subset S_0 \subset \dots S_{-p} \subset S_{-p-1} \subset \cdots \subset \bigcup_{p\in\mathbf{N}} S_p = S'.$$

We denote by $\langle\omega,\xi\rangle$ the dual pairing of $\omega \in S'$ and $\xi \in S$. For $\omega \in S_0$, we have $\langle\omega,\xi\rangle = (\omega,\xi)_0$. The space S is a countably Hilbert nuclear space endowed with the projective limit topology. Its dual S' is the inductive limit of $\{S_{-p},\ p \geqslant 1\}$.

Consider Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \quad n = 0,1,2,\dots$$

and the corresponding Hermite functions

$$\xi_n(x) = \frac{1}{\pi^{1/4}(n!)^{1/2}2^{n/2}}H_n(x)e^{-(x^2/2)}, \quad n = 0,1,2,\dots.$$

The set $\{\xi_n\}_{n=0}^{\infty}$ is an orthonormal basis for S_0 and we have

$$A\xi_n = (2n+2)\xi_n, \quad n = 0,1,2,\dots.$$

For any $\xi \in S_p$, $p \in \mathbb{Z}$ we have

$$|\xi|_p = \left(\sum_{n=0}^{\infty}(2n+2)^{2p}(\xi,\xi_n)_0^2\right)^{1/2}.$$

Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)_H$ and the corresponding norm $\|\cdot\|_H$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in H.

Consider tensor products of Hilbert spaces $S_p \otimes H$ for $p \in \mathbb{Z}$. Denote by $[\cdot,\cdot]_p$ the inner product in $S_p \otimes H$ and by $\|\cdot\|_p$ the corresponding norm. Since $\{\xi_i \otimes e_j\}_{i=0,j=1}^{\infty}$ is an orthonormal basis in $S_0 \otimes H$, any $\eta \in S_p \otimes H$ admits the following unique representation

$$\eta = \sum_{i=0;j=1}^{\infty}\eta_{ij}(\xi_i \otimes e_j) = \sum_{j=1}^{\infty}\eta_j \otimes e_j = \sum_{i=0}^{\infty}\xi_i \otimes h_i,$$

where $\eta_{ij} = [\eta, \xi_i \otimes e_j]_0$, $\eta_j = \sum_{i=0}^{\infty} \eta_{ij}\xi_i \in S_p$, $h_i = \sum_{j=1}^{\infty} \eta_{ij} e_j \in H$.

We have

$$\|\eta\|_p^2 = \sum_{i=0;j=1}^{\infty} \eta_{ij}^2 (2i+2)^{2p} = \sum_{j=1}^{\infty} |\eta_j|_p^2 = \sum_{i=0}^{\infty} (2i+2)^{2p} \|h_i\|_H^2 .$$

For the inner product in $S_p \otimes H$ we have

$$[\eta, \theta]_p = \sum_{i=0;j=1}^{\infty} \eta_{ij}\theta_{ij}(2i+2)^{2p} = \sum_{j=1}^{\infty} (\eta_j, \theta_j)_p^2 = \sum_{i=0}^{\infty} (2i+2)^{2p} (h_i, g_i)_H .$$

Consider tensor products $S \otimes H$ and $S' \otimes H$. We have

$$S \otimes H = \bigcap_{p \in \mathbf{N}} S_p \otimes H \subset \cdots \subset S_{p+1} \otimes H \subset S_p \otimes H \subset \cdots \subset S_0 \otimes H \subset$$
$$\subset \ldots S_{-p} \otimes H \subset S_{-p-1} \otimes H \subset \cdots \subset \bigcup_{p \in \mathbf{N}} S_p \otimes H = S' \otimes H .$$

Clearly, $S \otimes H$ is a countably Hilbert space endowed with the projective limit topology, $S' \otimes H$ is its dual and is the inductive limit of $\{S_{-p} \otimes H , \ p \geqslant 1\}$. Note that $S \otimes H$ is not a nuclear space.

Denote by $[\cdot, \cdot]$ the dual pairing of elements from $S' \otimes H$ and $S \otimes H$. For any $\omega \in S' \otimes H$ and $\eta \in S \otimes H$ with

$$\omega = \sum_{i=0;j=1}^{\infty} \omega_{ij}(\xi_i \otimes e_j) = \sum_{j=1}^{\infty} \omega_j \otimes e_j = \sum_{i=0}^{\infty} \xi_i \otimes g_i \ , \ \omega_{ij} \in \mathbb{R}, \omega_j \in S', g_i \in H$$

and

$$\eta = \sum_{i=0;j=1}^{\infty} \eta_{ij}(\xi_i \otimes e_j) = \sum_{j=1}^{\infty} \eta_j \otimes e_j = \sum_{i=0}^{\infty} \xi_i \otimes h_i \ , \ \eta_{ij} \in \mathbb{R}, \eta_j \in S, h_i \in H ,$$

we have

$$[\omega, \eta] = \sum_{i=0,j=1}^{\infty} \omega_{ij}\eta_{ij} = \sum_{j=1}^{\infty} \langle \omega_j, \eta_j \rangle = \sum_{i=0}^{\infty} (g_i, h_i)_H .$$

In particular, if $\omega \in S_0 \otimes H$, then $[\omega, \eta] = [\omega, \eta]_0$.

Now we numerate the elements of $\{\xi_i \otimes e_j\}_{i=0,j=1}^{\infty}$. Define $\varepsilon_k = \xi_i \otimes e_j$, where

$$k = k(i,j) = 1 + 2 + \cdots + (i+j-1) + j = \frac{(i+j)^2 + j - i}{2} .$$

In this case we have

$$j = j(k) = k - \frac{\mathcal{N}(k)(\mathcal{N}(k) - 1)}{2}$$

and

$$i = i(k) = \frac{\mathcal{N}(k)(\mathcal{N}(k)+1)}{2} - k\,,$$

where

$$\mathcal{N}(k) = \max\left\{n \in \mathbb{N} \;\Big|\; \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \leqslant k\right\}.$$

3. Q-WHITE NOISE MEASURE ON $S' \otimes H$

Let Q be a linear operator in H, defined by

$$Qx = \sum_{j=1}^{\infty} \sigma_j^2 (x, e_j)_H e_j, \quad x \in H$$

with $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$. It is positive, self-adjoint and nuclear.

Consider a functional on $S \otimes H$ defined by

$$C_Q(\eta) = \exp\left\{-\frac{1}{2}\left[(I \otimes Q)\eta, \eta\right]_0\right\}, \quad \eta \in S \otimes H\,.$$

Denote by $\mathfrak{B}$ the Borel σ-field in $S' \otimes H$.

THEOREM 1. *There exists a probability measure* $\mathfrak{m}_Q$ *on* $(S' \otimes H, \mathfrak{B})$ *such that*

$$C_Q(\eta) = \int_{S' \otimes H} \exp\{i[\omega, \eta]\} d\mathfrak{m}_Q(\omega)\,, \quad \eta \in S \otimes H\,.$$

PROOF: Denote by $P_{\varepsilon_1,\dots,\varepsilon_n}$ the projector from $S' \otimes H$ onto $Sp\{\varepsilon_1, \dots, \varepsilon_n\}$:

$$P_{\varepsilon_1,\dots,\varepsilon_n} : \omega = \sum_{k=1}^{\infty} \omega_{i(k),j(k)} \varepsilon_k \mapsto \sum_{k=1}^{n} \omega_{i(k),j(k)} \varepsilon_k\,.$$

Let $\rho_{\varepsilon_1,\dots,\varepsilon_n} : P_{\varepsilon_1,\dots,\varepsilon_n}(S' \otimes H) \to \mathbb{R}^n$ be the natural isomorphism. Denote by $\mathfrak{B}_{\varepsilon_1,\dots,\varepsilon_n}$ the collection of subsets in $S' \otimes H$ defined by $\mathfrak{B}_{\varepsilon_1,\dots,\varepsilon_n} = P^{-1}_{\varepsilon_1,\dots,\varepsilon_n} \rho^{-1}_{\varepsilon_1,\dots,\varepsilon_n}\left(\mathcal{B}(\mathbb{R}^n)\right)$, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-field in $\mathbb{R}^n$. It consists of all sets of the form

$$A = \left\{\omega = \sum_{k=1}^{\infty} \omega_{i(k),j(k)} \varepsilon_k \in S' \otimes H \mid \left(\omega_{i(1),j(1)}, \dots, \omega_{i(n),j(n)}\right) \in B\right\},\; B \in \mathcal{B}(\mathbb{R}^n)\,.$$

Define

$$C_{\varepsilon_1,\dots,\varepsilon_n}(\overline{z}) = C_Q(z_1\varepsilon_1 + \dots + z_n\varepsilon_n), \quad \overline{z} = (z_1, \dots, z_n) \in \mathbb{R}^n\,.$$

For any $n \in \mathbb{N}$, $C_{\varepsilon_1,\dots,\varepsilon_n}$ is a continuous positive-definite functional on $\mathbb{R}^n$ with $C_{\varepsilon_1,\dots,\varepsilon_n}(0) = 1$. Therefore by Bochner's theorem it is a characteristic functional of a probability measure $m_{\varepsilon_1,\dots,\varepsilon_n}$ on the measurable space $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$, so that

$$C_{\varepsilon_1,\dots,\varepsilon_n}(\overline{z}) = \int_{\mathbb{R}^n} \exp\{i(\overline{x}, \overline{z})\} dm_{\varepsilon_1,\dots,\varepsilon_n}(\overline{x})\,, \quad \overline{z} \in \mathbb{R}^n\,.$$

Let $\mathfrak{m}_{\varepsilon_1,\dots,\varepsilon_n}$ be a probability measure on $(S'\otimes H, \mathfrak{B}_{\varepsilon_1,\dots,\varepsilon_n})$ defined by

$$\mathfrak{m}_{\varepsilon_1,\dots,\varepsilon_n}(A) = m_{\varepsilon_1,\dots,\varepsilon_n}(B),\ A\in\mathfrak{B}_{\varepsilon_1,\dots,\varepsilon_n},\ A = P^{-1}_{\varepsilon_1,\dots,\varepsilon_n}\rho^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(B),\quad B\in\mathcal{B}(\mathbb{R}^n).$$

It is not difficult to see that $\{m_{\varepsilon_1,\dots,\varepsilon_n}\}_{n=1}^{\infty}$ is a consistent family of measures. Therefore by Kolmogorov's theorem there exists a probability space $(\Omega,\mathcal{F},P)$ and a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ such that

$$m_{\varepsilon_1,\dots,\varepsilon_n} = P(\overline{X}_n^{-1})\quad\text{with}\quad \overline{X}_n = (X_1,\dots,X_n),\ n=1,2,\dots,$$

and we have

$$\begin{aligned} C_{\varepsilon_1,\dots,\varepsilon_n}(\overline{z}) &= \int_{\mathbb{R}^n}\exp\{i(\overline{x},\overline{z})\}dm_{\varepsilon_1,\dots,\varepsilon_n}(\overline{x})\\ &= \int_{S'\otimes H}\exp\{i[\omega, z_1\varepsilon_1+\cdots+z_n\varepsilon_n]\}d\mathfrak{m}_{\varepsilon_1,\dots,\varepsilon_n}(\omega)\\ &= \int_{\Omega}\exp i(\overline{X}_n,\overline{z})\,dP. \end{aligned}\tag{3}$$

$\square$

Lemma 1. *For any $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that for any $p\in\mathbb{N}$*

$$\int_{\Omega}\exp\Big\{-\frac{1}{2}\sum_{k=k_0}^{\infty}(2i(k)+2)^{-2p}X_k^2\Big\}dP > 1-\varepsilon.$$

Proof: For any $m,l\in\mathbb{N}$ with $m<l$ we have

$$\begin{aligned} &\int_{\Omega}\exp\Big\{-\frac{1}{2}\sum_{k=m}^{l}(2i(k)+2)^{-2p}X_k^2\Big\}dP\\ &= \int_{\Omega}\int_{\mathbb{R}^{l-m}}\exp\Big\{i\sum_{k=m}^{l}X_kz_k\Big\}\frac{\prod_{k=m}^{l}(2i(k)+2)^p}{(2\pi)^{((l-m)/2)}}\exp\Big\{-\frac{1}{2}\sum_{k=m}^{l}(2i(k)+2)^{2p}z_k^2\Big\}d\overline{z}\,dP\\ &= \frac{\prod_{k=m}^{l}(2i(k)+2)^p}{(2\pi)^{((l-m)/2)}}\int_{\mathbb{R}^{l-m}}C_{\varepsilon_m,\dots,\varepsilon_l}(z_m,\dots,z_l)\exp\Big\{-\frac{1}{2}\sum_{k=m}^{l}(2i(k)+2)^{2p}z_k^2\Big\}d\overline{z}\\ &= \frac{1}{(2\pi)^{((l-m)/2)}}\int_{\mathbb{R}^{l-m}}C_{\varepsilon_m,\dots,\varepsilon_l}\Big(\frac{z_m}{(2i(m)+2)^p},\dots,\frac{z_l}{(2i(l)+2)^p}\Big)\exp\Big\{-\frac{1}{2}\sum_{k=m}^{l}z_k^2\Big\}d\overline{z}. \end{aligned}$$

Therefore

$$1-\int_{\Omega}\exp\Big\{-\frac{1}{2}\sum_{k=m}^{l}(2i(k)+2)^{-2p}X_k^2\Big\}dP$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{((l-m)/2)}} \int_{\mathbb{R}^{l-m}} \left(1 - C_{\varepsilon_m,\ldots,\varepsilon_l}\left(\frac{z_m}{(2i(m)+2)^p}, \ldots, \frac{z_l}{(2i)(l)+2)^p}\right)\right) \exp\left\{-\frac{1}{2}\sum_{k=m}^{l} z_k^2\right\} d\overline{z} \\
&= \frac{1}{(2\pi)^{((l-m)/2)}} \int_{\mathbb{R}^{l-m}} \left(1 - \exp\left\{-\frac{1}{2}\sum_{k=m}^{l} \frac{\sigma_{j(k)}^2 z_k^2}{(2i(k)+2)^{2p}}\right\}\right) \exp\left\{-\frac{1}{2}\sum_{k=m}^{l} z_k^2\right\} d\overline{z} \\
&\leqslant \frac{1}{(2\pi)^{((l-m)/2)}} \int_{\mathbb{R}^{l-m}} \sum_{k=m}^{l} \frac{\sigma_{j(k)}^2 z_k^2}{(2i(k)+2)^{2p}} \exp\left\{-\frac{1}{2}\sum_{k=m}^{l} z_k^2\right\} d\overline{z} \\
&= \sum_{k=m}^{l} \frac{\sigma_{j(k)}^2 z_k^2}{(2i(k)+2)^{2p}} = \sum_{k=m}^{l} \frac{\sigma_{j(k)}^2 z_k^2}{(2i(k)+2)^{2}}.
\end{aligned}
$$

Since

$$\sum_{k=1}^{\infty} \frac{\sigma_{j(k)}^2 z_k^2}{(2i(k)+2)^2} = \sum_{j=1}^{\infty} \sigma_j^2 \sum_{i=1}^{\infty} \frac{1}{(2i+2)^2} < \infty$$

as a product of absolutely convergent series, we let $l \to \infty$ and apply the Lebesgue dominated convergence theorem. We have

$$1 - \int_{\Omega} \exp\left\{-\frac{1}{2}\sum_{k=m}^{\infty}(2i(k)+2)^{-2p} X_k^2\right\} dP \leqslant \sum_{k=m}^{\infty} \frac{\sigma_{j(k)}^2}{(2i(k)+2)^2}.$$

Hence the assertion follows.

End of the Proof of Theorem 1. Given $\varepsilon > 0$ we use Lemma 1 to choose $m \in \mathbb{N}$ so that for any $p \in \mathbb{N}$

$$
\begin{aligned}
P\left\{\sum_{k=1}^{\infty}(2i(k)+2)^{-2p} X_k^2 < \infty\right\} &= \int_{\{\sum_{k=m}^{\infty}(2i(k)+2)^{-2p} X_k^2 < \infty\}} 1\, dP \\
&\geqslant \int_{\{\sum_{k=m}^{\infty}(2i(k)+2)^{-2p} X_k^2 < \infty\}} \exp\left\{-\frac{1}{2}\sum_{k=m}^{\infty}(2i(k)+2)^{-2p} X_k^2\right\} dP \geqslant 1 - \varepsilon.
\end{aligned}
$$

Hence

$$P\left\{\sum_{k=1}^{\infty}(2i(k)+2)^{-2p} X_k^2 < \infty\right\} = 1.$$

Define

$$X(\omega) = \sum_{k=m}^{\infty} X_k(\omega)\varepsilon_k, \quad \omega \in \Omega.$$

The mapping $X : \Omega \to S' \otimes H$ is measurable. Let $\mathfrak{m}_Q = P \circ X^{-1}$. It is a probability Borel measure on $S' \otimes H$.

By (3) we have

$$C_Q(P_{\varepsilon_1,\ldots,\varepsilon_n}\eta) = \int_{\Omega} \exp\{i[P_{\varepsilon_1,\ldots,\varepsilon_n} X, \eta]\}\, dP.$$

Since $P_{\varepsilon_1,\dots,\varepsilon_n}\eta \to \eta$ as $n \to \infty$ in $S \otimes H$ and C_Q is continuous, by Lebesgue's dominated convergence theorem we have

$$\int_\Omega \exp\{i[P_{\varepsilon_1,\dots,\varepsilon_n}X,\eta]\}\,dP \longrightarrow \int_\Omega \exp\{i[X,\eta]\}\,dP\,, \quad n \to \infty\,.$$

Hence we obtain

$$C_Q(\eta) = \int_\Omega \exp\{i[X,\eta]\}\,dP = \int_{S'\otimes H} \exp\{i[\omega,\eta]\}\,d\mathfrak{m}(\omega)\,. \qquad \square$$

REMARK. Note that $\mathfrak{m}_Q(S_{-p}\otimes H) = 1$ for any $p \geqslant 1$. Hence, $\mathfrak{m}_Q$ is supported by $S_{-1}\otimes H$.

4. Q-WHITE NOISE MEASURE ON $S'(H)$

Consider the space $S'(H)$ of H-valued distributions. It consists of all linear continuous operators from S to H. We write $\omega(\xi)$ for $\omega \in S'(H)$ evaluated against $\xi \in S$. For any $\omega = \sum_{j=1}^{\infty} \omega_j \otimes e_j \in S' \otimes H$ we define $J\omega \in S'(H)$ by

$$J\omega(\xi) = \sum_{j=1}^{\infty} \langle \omega_j, \xi\rangle e_j\,, \quad \xi \in S\,. \tag{4}$$

Since the mapping $J : S' \otimes H \to S'(H)$ is an isomorphism, we identify $\omega \in S' \otimes H$ with $J\omega \in S'(H)$ and use the same notation. So we write

$$\omega(\xi) = \Big(\sum_{j=1}^{\infty} \omega \otimes e_j\Big)(\xi) = \sum_{j=1}^{\infty} \langle \omega, \xi\rangle e_j\,.$$

Denote by $\mathcal{B}$ the σ-field in $S'(H)$ defined by $\mathcal{B} = J(\mathfrak{B})$. Obviously $\mathcal{B}$ coincides with the Borel σ-field in $S'(H)$. For any $A \in \mathcal{B}$ let $\mu_Q(A) = \mathfrak{m}_Q(B)$ where B satisfies $A = J(B)$.

Let $\omega \in S'(H), \xi \in S, h = \sum_{j=1}^{\infty} h_j e_j \in H$. Then we have

$$\big(\omega(\xi), h\big)_H = \bigg(\Big(\sum_{j=1}^{\infty} \omega \otimes e_j\Big)(\xi), h\bigg)_H = \sum_{j=1}^{\infty} \langle \omega_j, \xi\rangle h_j = \sum_{j=1}^{\infty} \langle \omega_j, h_j\xi\rangle = [\omega, \xi_h]\,.$$

Here $\xi_h = \sum_{j=1}^{\infty} h_j \xi \otimes e_j \in S \otimes H$ since for any $p \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} |h_j\xi|_p^2 = |\xi|_p^2 \sum_{j=1}^{\infty} h_j^2 < \infty\,.$$

Hence the following equality holds true

$$\int_{S'(H)} \exp\Big\{i\big(\omega(\xi), h\big)_H\Big\}\, d\mu_Q(\omega) = \int_{S'\otimes H} \exp\{i[\omega, \xi_h]\} d\mathrm{m}_Q(\omega)$$
$$= \exp\Big\{-\frac{1}{2}\big[(I\otimes Q)\xi_h, \xi_h\big]_0\Big\} = \exp\Big\{-\frac{1}{2}\sum_{j=1}^{\infty}\sigma_j^2 |h_j \xi|_0^2\Big\}$$
$$= \exp\Big\{-\frac{1}{2}|\xi|_0^2 (Qh, h)_H\Big\}. \tag{5}$$

Consider the probability space $(S'(H), \mathcal{B}, \mu_Q)$. Define a generalised H-valued stochastic process $\{\mathbb{W}(\xi, \omega),\ \xi \in S\}$ by

$$\mathbb{W}(\xi, \omega) = \omega(\xi).$$

It follows from the equality (5) that for any $h \in H$ the $\mathbb{R}$-valued generalised stochastic process $\{(\mathbb{W}(\xi, \omega), h)_H,\ \xi \in S\}$, which can be regarded as a projection of $\mathbb{W}$ onto $\mathrm{Sp}\{h\}$, is a smoothed white noise with variance $(Qh, h)_H$. On the other hand, for any $\xi \in S$, $\mathbb{W}(\xi, \cdot)$ is an H-valued Gaussian random variable with mean 0 and covariance operator $|\xi|_0^2 Q$. Therefore we refer to $(S'(H), \mathcal{B}, \mu_Q)$ as *the H-valued Q-white noise space*. The generalised stochastic process $\mathbb{W}(\xi, \omega)$ is referred to as *the H-valued Q-white noise*.

Consider the space $L^2(S'(H); H)$ of square (Bochner) integrable H-valued random variables defined on $S'(H)$. For any $\xi \in S$ random variable $\mathbb{W}(\xi, \cdot) : S'(H) \to H$ belongs to $L^2(S'(H); H)$. We have

$$\big\|\mathbb{W}(\xi, \cdot)\big\|^2_{L^2(S'(H);H)} = \operatorname{Tr} Q \cdot \|\xi\|^2_{S_0}. \tag{6}$$

Define stochastic process $\{W(t) \mid t \geqslant 0\}$ on $(S'(H), \mathcal{B}, \mu_Q)$ by

$$W(t)(\omega) = \omega(\chi_{[0;t]}) := \lim_{n\to\infty} \omega(\theta_n), \tag{7}$$

where limit is taken in $L^2(S'(H); H)$ and $\{\theta_n\}_{n=1}^{\infty} \subset S$ is a sequence convergent to $\chi_{[0;t]}$ in $L^2(\mathbb{R})$. Existence of the limit in (7) and its independence of the choice of $\{\theta_n\}_{n=1}^{\infty} \subset S$ follow from (6). It is not difficult to check that $W(t)$ is a Q-Wiener process. Its trajectories are continuous H-valued functions.

For any $\xi \in S$ we have

$$-\int_{\mathbb{R}} W(t)\xi'(t)dt = -\int_{\mathbb{R}} \omega\big(\chi_{[0;t]}\big)\xi'(t)\, dt = \omega\Big(-\int_{\mathbb{R}} \chi_{[0;t]}(s)\xi'(t)\, dt\Big)$$
$$= \omega\Big(-\int_s^{\infty} \xi'(t)\, dt\Big) = \omega(\xi).$$

Thus, $\mathbb{W}$ can be regarded as a generalised derivative of $W(t)$ (in $S'(H)$ sense).

Let $W_0(t)$ be defined by

$$W_0(t) = \begin{cases} W(t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Its trajectories are continuous with probability 1. Define generalised stochastic process $\mathbb{W}_0$ by $\mathbb{W}_0(\xi, \omega) = W_0'(\xi)$, where derivative is understood in the generalised sense:

$$\mathbb{W}_0(\xi, \omega) = -\int_{\mathbb{R}} W_0(t)\xi'(t)\, dt = -\int_0^\infty W(t)\xi'(t)\, dt.$$

It is natural to call $\mathbb{W}_0$ the Q-white noise with support in $[0, \infty)$, or the Q-white noise starting at $t = 0$.

5. Equations with additive noise

Let X, Y and H be separable Hilbert spaces. Consider the equation

$$P * U = F + B\mathbb{W}_0, \tag{8}$$

where $P \in D'_+\big(\mathcal{L}(X;Y)\big)$, $U \in D'_+(X)$, $F \in D'_+(Y)$, $B \in \mathcal{L}(H;Y)$ and $\mathbb{W}_0$ is the H-valued Q-white noise with support in $[0;\infty)$, on the probability space $(S'(H), \mathcal{B}, \mu_Q)$. Let P have a convolution inverse $G \in D'_+\big(\mathcal{L}(Y;X)\big)$. Then the generalised stochastic process $\big\{U(\xi,\omega)\,,\ \xi \in S\big\}$, defined by

$$U(\xi, \omega) := (G * F)(\xi) + (G * B\mathbb{W}_0)(\xi, \omega), \tag{9}$$

is the unique solution of (8). Convolution $G * B\mathbb{W}_0$ is well defined since $B\mathbb{W}_0(\cdot, \omega)$ has support bounded from below for any $\omega \in S'(H)$ (see [**1**]).

Now we consider a particular example of P. Let A be a closed linear operator acting in Y and $X = [D(A)]$ be the domain of A, endowed with the graph-norm. Then

$$P = \delta' \otimes I - \delta \otimes A \in D'_+\big(\mathcal{L}(X;Y)\big).$$

Define $F \in D'_+(Y)$ by

$$F(\xi) := \xi(0)\, u^0 + \int_0^\infty \xi(t) f(t)\, dt\,, \quad \xi \in D\,,\ f \in L_1^{\mathrm{loc}}(\mathbb{R}, Y)\,,\ u^0 \in Y. \tag{10}$$

Then the Cauchy problem

$$u'(t) = Au(t) + f(t)\,, \quad t > 0\,, \quad u(0) = u^0 \tag{11}$$

can be written in the form

$$P * U = F$$

(see [1, 4]). If the right-hand side of (11) is perturbed by a white noise term, then it is natural to write it in form (8) in the space of distributions $S'(H)$.

Let A in (10) be the generator of a C_0-semigroup $\{S(t),\ t \geqslant 0\}$. Then the convolution inverse to P is

$$G(\xi) = \int_0^\infty \xi(t)S(t)\,dt\,,$$

and formula (9) becomes

$$U(\xi,\omega) = \int_0^\infty \xi(t)S(t)u^0\,dt + \int_0^\infty \int_0^t S(t-s)f(s)\,ds\,\xi(t)\,dt \\ - \int_0^\infty \int_0^t S(t-s)B\omega(\chi_{[0;s]})\,ds\,\xi(t)\,dt\,.$$

If A is the generator of an exponentially bounded n-times integrated semigroup $\{V(t),\ t \geqslant 0\}$, then the convolution inverse to P has the form

$$G(\xi) = (-1)^n \int_0^\infty \xi^{(n)}(t)V(t)\,dt\,,$$

and formula (9) becomes

$$U(\xi,\omega) = (-1)^n \int_0^\infty \xi^{(n)}(t)V(t)u^0\,dt + (-1)^n \int_0^\infty \int_0^t V(t-s)f(s)\,ds\,\xi^{(n)}(t)\,dt \\ + (-1)^{n+1} \int_0^\infty \int_0^t V(t-s)B\omega(\chi_{[0;s]})\,ds\,\xi^{(n+1)}(t)\,dt\,.$$

References

[1] H.O. Fattorini, *The Cauchy Problem* (Cambridge University Press, Cambridge, 1983).

[2] T. Hida, *Brownian motion*, Applications of Mathematics **11** (Springer-Verlag, Berlin, Heidelberg, New York, 1980).

[3] H. Kuo, *Gaussian Measures in Banach Spaces*, Springer Lecture Notes in Maths. **643** (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[4] I. Melnikova and A. Filinkov, *Abstract Cauchy problems: Three approaches* (Chapman & Hall, Boca Raton, 2001).

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